XYX Algebras

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Algebras Generated by Graphs

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- Given a graph G, we construct an algebra associated with it.
- Each vertex represents a generator
- Generators follow certain relations based on whether the corresponding vertices are adjacent.

XYX Algebras

In an XYX algebra, the following relations hold:

- If two generators, x and y are not adjacent, then they commute ($xy = yx$).
- If two generators, x and y are adjacent, then $xyx = 0$ and $yxy = 0$.
- The square of any generator is zero ($xx = 0$).

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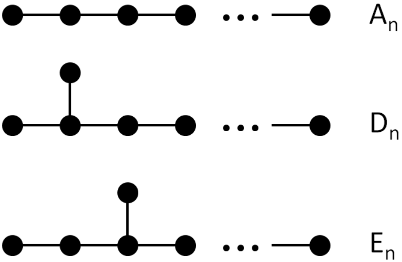
Conjecture

The only graphs corresponding to finite XYX algebras are the Dynkin diagrams A_n, D_n, E_6, E_7, and E_8.



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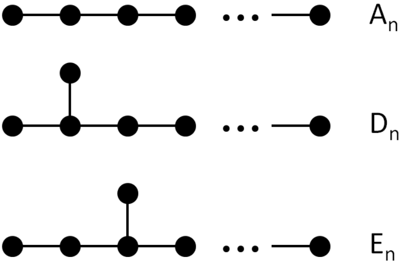

This suggests a natural connection between Coxeter groups and XYX algebras.

Examples of Graphs with Infinite Algebras

We have identified several conditions on the graph that make the algebra infinite:

- Cycle
- Vertex of degree 4
- Two or more vertices of degree 3

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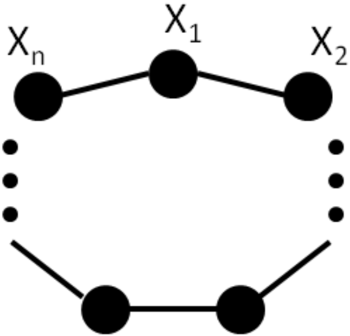
This limits all graphs with a finite algebra to either the path graphs or graphs with one vertex that has three paths branching out of it.

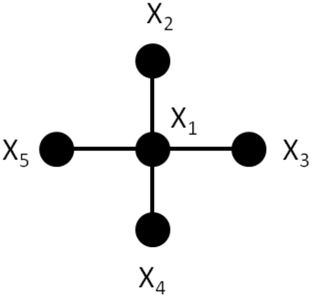
Cycle Graphs

- If we have a cycle made with generators $x_1, x_2, ..., x_n$, then we can have an infinite number of words of the form $x_1 x_2 ... x_n x_1 .....$



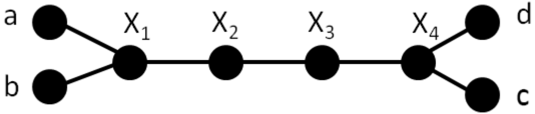
Vertex of Degree 4

- If we have a vertex of degree 4, then we can form an infinite number of words by repeating $x_1x_2x_3x_1x_4x_5x_1x_2x_3x_1x_4x_5$



Two or More Vertices of Degree 3

- If we have two or more vertices of degree 3, then we can form an infinite number of words by repeating $abx_1x_2x_3x_4cdx_4x_3x_2x_1abx_1x_2x_3x_4cdx_4x_3x_2x_1$



Enumeration of Finite Graphs

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- We started with the path graph, also known as the Dynkin diagram for A_n.

A_n

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A_n

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- The dimensions were 1, 2, 5, 14, and 42, which are Catalan numbers.

Theorem

The dimensions of the XYX algebras of the path graphs are the Catalan numbers.

Catalan Numbers

- One of the most common sequences in mathematics
- Over 150 definitions
- One such definition is the Dyck paths

Dyck Paths

- Dyck paths are a way of representing the Catalan numbers as a set of paths between two points.
- Below is a sample Dyck path of length 8.

A_n

- We counted the words in the algebra by length.

A_n

- We counted the words in the algebra by length.
- We discovered an interesting connection to Dyck paths.

Theorem

The number of words of a given length in the XYX algebra of a path graph is the same as the number of Dyck paths where the sum of the heights of the peaks minus the number of peaks is the length of the word.

A_n

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A_n

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- We build the bijection as follows.

A_n

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- Write each run as a pair (p_i, r_i), where p_i is the first generator of the run and r_i is the length of the run.

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- Write each run as a pair (p_i, r_i), where p_i is the first generator of the run and r_i is the length of the run.
- Draw a mountain top with peak coordinates $(p_i + r_i + 1, r_i + 1)$.

A_n

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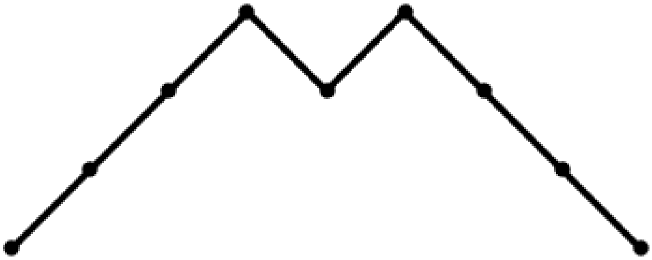
- Write each run as a pair (p_i, r_i), where p_i is the first generator of the run and r_i is the length of the run.
- Draw a mountain top with peak coordinates $(p_i + r_i + 1, r_i + 1)$.
- Superimpose all the mountains.
- Add trivial peaks if necessary.

Examples

- Here is the sample Dyck path corresponding to $x_2x_1x_3x_2$.
- The two peaks correspond to the descending runs x_2x_1 and x_3x_2



Future Directions

There are many future directions we could take this project.

- Finish our proof that only the Dynkin diagrams of finite Coxeter groups produce a finite XYX algebra
- Enumerate the dimensions on algebras corresponding to other graphs, such as D_n
- Enumerate the number of words of a given length in an infinite algebra

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